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MINIMUM COVERING GUTMAN ENERGY OF A GRAPH

Roshan Sara Philipose
Research Scholar, Noorul Islam University, Kumaracoil, Tamil Nadu, India
roshanjilu@gmail.com

Sarasija P.B
Professor, Noorul Islam University, Kumaracoil, Tamil Nadu, India
sijavk@gmail.com

Abstract

The concept of a new kind of graph energy, namely, minimum covering energy, denoted by $E_c(G)$, was introduced by Chandrashekar Adiga et.al in 2012. The Gutman energy is the sum of the absolute values of the eigenvalues obtained from the Gutman matrix. In this paper, we depict the minimum covering Gutman energy of a graph which can be defined as sum of the absolute values of the minimum covering Gutman eigenvalues obtained from the minimum covering Gutman matrix of a graph, $A_{c_g^{'}}(G) := (g_{ij})$, *where* $g_{ij} = \begin{cases} 1, & if\ i = j \ \&\ v_i \in C' \\ 0, & if\ i = j \ \&\ v_i \notin C' \\ d_i d_j d_G(v_i, v_j), & otherwise \end{cases}$ *. Here,* d_i *is the degree of the node* v_i, $d_G(v_i, v_j)$ *is the shortest distance between the nodes* v_i & v_j *and* C' *is the minimum covering set. Further, we establish the upper and lower bounds for minimum covering Gutman energy.*

Keywords

Minimum Covering Gutman Energy, Minimum Covering Gutman Matrix, Minimum Covering Set, Upper Bound

1. Introduction

Throughout the paper, we take account of a simple connected graph G with node set V containing q nodes and edge set E containing p edges. For the detailed study of graphs and matrices, view (Bapat, 2011). Since we aim at determining the minimum covering Gutman energy of a graph, we should need a brief analysis on minimum covering Gutman matrix.

To start with, let $v_1, v_2, \ldots, v_q$ be the nodes of G. A covering set C' of G can be defined as a subset of V in which atleast one node of C' must be incident with every edge of G and any covering set with minimum cadinality is termed as minimum covering set.

Now, we define the minimum covering Gutman Matrix

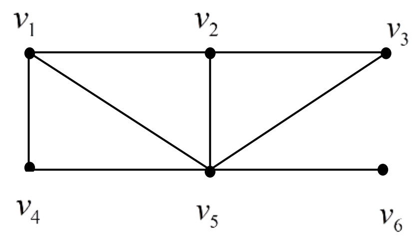
$A_{C'_g}(G) := (g_{ij})$, $g_{ij} = \begin{cases} 1, & if\ i = j \ \&\ v_i \in C' \\ 0, & if\ i = j \ \&\ v_i \notin C' \\ d_i d_j d_G(v_i, v_j), & otherwise \end{cases}$, where d_i, d_j and $d_G(v_i, v_j)$ denote the degree of node v_i, degree of node v_j and the shortest distance between the nodes v_i and v_j respectively. Note that we shall take here. Refer both references (Roshan et al., 2018) for the detailed study of illustration of Gutman index and Gutman matrix. Then the minimum covering Gutman eigenvalues are the eigenvalues $\eta_1, \eta_2, \ldots, \eta_q$ obtained from the characteristic polynomial, $p_q(G, \eta) = \det(\eta I - A_{C'_g}(G))$. Obviously, they are real as $A_{C'_g}(G)$ is real, symmetric and they are labeled in non-increasing order $\eta_1 \geq \eta_2 \geq \ldots \geq \eta_q$. So, the minimum covering Gutman energy, denoted by $GE_{C'}(G)$, is the sum of the absolute values of the minimum covering Gutman eigenvalues. *i.e.*, $GE_{C'}(G) = \sum_{i=1}^{q} |\eta_i|$. See (Gutman et al., 1978, Balakrishnan, 2004) for the study of energy of graphs and (Adiga et al., 2012) for minimum covering energy of graphs

Rajesh Kanna et.al determined minimum covering distance energy of a graph that motivates us to mould this paper with the ideas of minimum covering Gutman energy of a graph (Rajesh Kanna et al., 2013). We have sectioned this paper into four. Following by introduction in section 1, we are trying to convince the method of determining minimum covering set and thus finding the minimum covering Gutman energy through an example in section 2. In section 3, we are finding the minimum covering Gutman energy of some standard graphs – Cocktail party graph, Star graph and Crown graph. Finally in section 4, we are trying to establish some bounds of minimum covering Gutman energy of a graph.

2. Example of finding Minimum Covering Gutman Energy of a given Graph

2.1 Example:

G:



Solution: The possible minimum covering sets are $(i) C_1^{'} = \{v_1, v_2, v_5\}$, $(ii) C_2^{'} = \{v_2, v_4, v_5\}$ and $(iii) C_3^{'} = \{v_1, v_3, v_5\}$.

Now, we can find out the corresponding minimum covering Gutman matrix, characteristic equation, minimum covering Gutman eigenvalues and minimum covering Gutman energy for $C_1^{'}$.

$$(i)\ A_{C_{1_g}^{'}}(G) = \begin{pmatrix} 1 & 9 & 12 & 6 & 15 & 6 \\ 9 & 1 & 6 & 12 & 15 & 6 \\ 12 & 6 & 0 & 8 & 10 & 4 \\ 6 & 12 & 8 & 0 & 10 & 4 \\ 15 & 15 & 10 & 10 & 1 & 5 \\ 6 & 6 & 4 & 4 & 5 & 0 \end{pmatrix}.$$

Characteristic equation is

$$\eta^6 - 3\eta^5 - 1281.00153\eta^4 - 24446.05546\eta^3 - 167352.7414\eta^2 - 441244.116\eta - 388309.6854 = 0$$

Therefore, minimum covering Gutman eigenvalues are $\eta_1 \approx -16.3417$, $\eta_2 \approx -14$, $\eta_3 \approx -7.2977$, $\eta_4 \approx -2.5721$, $\eta_5 \approx -2$ and $\eta_6 \approx 45.2115$. Consequently, the minimum covering Gutman energy of the given graph G, $GE_{C_1^{'}}(G) = \sum_{i=1}^{6} |\eta_i| = 87.423$.

Similarly, we can find the minimum covering Gutman energy corresponding to $C_2^{'}$ and $C_3^{'}$. Notice that the minimum covering Gutman energy is depending on the covering set.

3. Minimum Covering Gutman Energy of some particular graphs

In this section, we shall consider some standard graphs and discuss their minimum covering Gutman energy.

3.1 Cocktail Party Graph

Definition 3.1.1: The Cocktail Party Graph ($K_{q\times 2}$) is a graph with node set V containing the union of $\{u_i, v_i\}$, where $i = 1,2,\ldots,q$ and edge set E containing the union of $\{u_i u_j, v_i v_j; i \neq j\}$ and $\{u_i v_j; 1 \leq i < j \leq q\}$.

Theorem 3.1.1: The minimum covering Gutman energy of Cocktail party graph $K_{q\times 2}$ is $16q(q-1)^2$.

Proof: We have the Cocktail Party Graph $K_{q\times 2}$ with node set $V = \bigcup_{i=1}^{q} \{u_i, v_i\}$ and edge set $E = \{u_i u_j, v_i v_j; i \neq j\} \bigcup \{u_i v_j, v_i u_j; 1 \leq i < j \leq q\}$. Here, the minimum covering set is $C' = \bigcup_{i=1}^{q-1} \{u_i, v_i\}$. Then the minimum covering Gutman matrix is given by $A_{C'_g}(K_{q\times 2}) =$

$$\begin{pmatrix}
1 & 8(q-1)^2 & 4(q-1)^2 & 4(q-1)^2 & \cdots & 4(q-1)^2 & 4(q-1)^2 & 4(q-1)^2 & 4(q-1)^2 \\
8(q-1)^2 & 1 & 4(q-1)^2 & 4(q-1)^2 & \cdots & 4(q-1)^2 & 4(q-1)^2 & 4(q-1)^2 & 4(q-1)^2 \\
4(q-1)^2 & 4(q-1)^2 & 1 & 8(q-1)^2 & \cdots & 4(q-1)^2 & 4(q-1)^2 & 4(q-1)^2 & 4(q-1)^2 \\
4(q-1)^2 & 4(q-1)^2 & 8(q-1)^2 & 1 & \cdots & 4(q-1)^2 & 4(q-1)^2 & 4(q-1)^2 & 4(q-1)^2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
4(q-1)^2 & 4(q-1)^2 & 4(q-1)^2 & 4(q-1)^2 & \cdots & 1 & 8(q-1)^2 & 4(q-1)^2 & 4(q-1)^2 \\
4(q-1)^2 & 4(q-1)^2 & 4(q-1)^2 & 4(q-1)^2 & \cdots & 8(q-1)^2 & 1 & 4(q-1)^2 & 4(q-1)^2 \\
4(q-1)^2 & 4(q-1)^2 & 4(q-1)^2 & 4(q-1)^2 & \cdots & 4(q-1)^2 & 4(q-1)^2 & 0 & 8(q-1)^2 \\
4(q-1)^2 & 4(q-1)^2 & 4(q-1)^2 & 4(q-1)^2 & \cdots & 4(q-1)^2 & 4(q-1)^2 & 8(q-1)^2 & 0
\end{pmatrix}$$

Therefore, the characteristic equation is

$$\left[\eta + 8(q-1)^2\right]\left[\eta + \left(8(q-1)^2 - 1\right)\right]^{q-1}[\eta - 1]^{q-2}\left[\eta^2 - \left(8q(q-1)^2 + 1\right)\eta + 8(q-1)^2\right] = 0.$$

So the minimum covering Gutman eigenvalues are $\eta = -8(q-1)^2$ (one time), $\eta = -\left(8(q-1)^2 - 1\right)$ ($(q-1)$ times), $\eta = 1$ ($(q-2)$ times) and

$$\eta = \frac{\left(8q(q-1)^2 + 1\right) \pm \sqrt{\left[8q(q-1)^2 + 1\right]^2 - 32(q-1)^2}}{2}$$ (one time each).

Thus the minimum covering Gutman energy is given by $GE_{C'}\left(K_{q\times 2}\right)=\left|-8(q-1)^2\right|+$

$$\left|-\left[8(q-1)^2-1\right](q-1)\right|+\left|1(q-2)\right|+\left|\frac{\left[8q(q-1)^2+1\right]\pm\sqrt{\left[8q(q-1)^2+1\right]^2-32(q-1)^2}}{2}\right|$$

$$=8(q-1)^2+\left(8(q-1)^2-1\right)(q-1)+q-2+8q(q-1)^2+1$$

$$=16q(q-1)^2$$

3.2 Star Graph

A star graph is the complete bipartite graph $K_{1,q-1}$.

Theorem 3.2.1: The minimum covering Gutman energy of star graph $K_{1,q-1}$ is $2(q-2)+\sqrt{4q^3-8q^2-8q+21}$, $q\geq 3$.

Proof: Let $v_0,v_1,\ldots,v_{q-1}$ be the nodes of $K_{1,\,q-1}$ and $C'=\{v_0\}$ be the minimum covering set. Its minimum covering Gutman matrix is given by

$$A_{C'_g}\left(K_{1,q-1}\right)=\begin{pmatrix} 1 & q-1 & q-1 & \cdots & q-1 & q-1 & q-1 \\ q-1 & 0 & 2 & \cdots & 2 & 2 & 2 \\ q-1 & 2 & 0 & \cdots & 2 & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ q-1 & 2 & 2 & \cdots & 0 & 2 & 2 \\ q-1 & 2 & 2 & \cdots & 2 & 0 & 2 \\ q-1 & 2 & 2 & \cdots & 2 & 2 & 0 \end{pmatrix}, \qquad q\geq 3.$$

Then for $q\geq 3$, its characteristic equation is $(\eta+2)^{q-2}\left(\eta^2-(2q-3)\eta-\left(q^3-3q^2+q+3\right)\right)=0$. So, the minimum covering Gutman eigenvalues are $\eta=-2$ ($(q-2)$ times) and $\eta=\dfrac{(2q-3)\pm\sqrt{(2q-3)^2+4\left(q^3-3q^2+q+3\right)}}{2}$ (one time each). Hence its minimum covering Gutman energy is given by

$$GE_{C'}(K_{1,q-1})=\left|-2(q-2)\right|+\left|\frac{(2q-3)\pm\sqrt{(2q-3)^2+4\left(q^3-3q^2+q+3\right)}}{2}\right|$$

$$=2(q-2)+\sqrt{(2q-3)^2+4\left(q^3-3q^2+q+3\right)}$$

$$=2(q-2)+\sqrt{4q^3-8q^2-8q+21}.$$

3.3 Crown Graph

Definition 3.3.1: For an integer $q \geq 2$, a Crown graph, denoted by S_q^0, is a graph with two sets of nodes $\{u_i^{'}; 1 \leq i \leq q\}$, $\{v_j^{'}; 1 \leq j \leq q\}$ and form an edge from $u_i^{'}$ to $v_j^{'}$ when $i \neq j$.

Theorem 3.3.1: The minimum covering Gutman energy of a Crown graph ($S_q^0, q \geq 2$) is

$$\begin{cases} 4.472, & for \quad q = 2 \\ 2\sqrt{257} + \sqrt{1601}, & for \quad q = 3 \\ \left(\sqrt{(2q-3)^2(2q-1)^2 + 8(q-1)^2}\right)(q-1) + \left(4(q-1)^3 + 1\right), & for \quad q \geq 4 \end{cases}$$

Proof: Let $\{u_1^{'}, u_2^{'}, \ldots, u_q^{'}, v_1^{'}, v_2^{'}, \ldots, v_q^{'}\}$ be the node set and $\{u_i^{'}v_j^{'}; 1 \leq i, j \leq q, i \neq j\}$ be the edge set of S_q^0, $q \geq 2$. Also, let $C^{'} = \{u_1^{'}, u_2^{'} \ldots, u_q^{'}\}$ be the minimum covering set.

Then its minimum covering Gutman matrix is given by $A_{C_g^{'}}\left(S_q^0\right) =$

$$\begin{pmatrix}
1 & 2(q-1)^2 & 2(q-1)^2 & \cdots & 2(q-1)^2 & 3(q-1)^2 & (q-1)^2 & (q-1)^2 & \cdots & (q-1)^2 \\
2(q-1)^2 & 1 & 2(q-1)^2 & \cdots & 2(q-1)^2 & (q-1)^2 & 3(q-1)^2 & (q-1)^2 & \cdots & (q-1)^2 \\
2(q-1)^2 & 2(q-1)^2 & 1 & \cdots & 2(q-1)^2 & (q-1)^2 & (q-1)^2 & 3(q-1)^2 & \cdots & (q-1)^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
2(q-1)^2 & 2(q-1)^2 & 2(q-1)^2 & \cdots & 1 & (q-1)^2 & (q-1)^2 & (q-1)^2 & \cdots & 3(q-1)^2 \\
3(q-1)^2 & (q-1)^2 & (q-1)^2 & \cdots & (q-1)^2 & 0 & 2(q-1)^2 & 2(q-1)^2 & \cdots & 2(q-1)^2 \\
(q-1)^2 & 3(q-1)^2 & (q-1)^2 & \cdots & (q-1)^2 & 2(q-1)^2 & 0 & 2(q-1)^2 & \cdots & 2(q-1)^2 \\
(q-1)^2 & (q-1)^2 & 3(q-1)^2 & \cdots & (q-1)^2 & 2(q-1)^2 & 2(q-1)^2 & 0 & \cdots & 2(q-1)^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
(q-1)^2 & (q-1)^2 & (q-1)^2 & \cdots & 3(q-1)^2 & 2(q-1)^2 & 2(q-1)^2 & 2(q-1)^2 & \cdots & 0
\end{pmatrix}$$

We can write its characteristic equation as follows:

$$\left[\eta^2 + (2q-3)(2q-1)\eta - 2(q-1)^2\right]^{q-1}\left[\eta^2 - \left(4(q-1)^3 + 1\right)\eta + (q-1)^3\left(3q(q-1)(q-4) + 2\right)\right] = 0, \; q \geq 3$$

So, the corresponding eigenvalues obtaining are

$$\eta = \frac{-(2q-3)(2q-1) \pm \sqrt{(2q-3)^2(2q-1)^2 + 8(q-1)^2}}{2}$$ (q-1 times each) and

$$\eta = \frac{\left[4(q-1)^3 + 1\right] \pm \sqrt{\left[4(q-1)^3 + 1\right]^2 - 4(q-1)^3\left[3q(q-1)(q-4) + 2\right]}}{2}$$ (one time each) .

Hence the minimum covering Gutman energy of S_q^0, $q \geq 4$, is given by

$$GE_{C'}\left(S_q^0\right)=\left|\left(\frac{-(2q-3)(2q-1)\pm\sqrt{(2q-3)^2(2q-1)^2+8(q-1)^2}}{2}\right)(q-1)\right|+$$

$$\left|\frac{\left[4(q-1)^3+1\right]\pm\sqrt{\left[4(q-1)^3+1\right]^2-4(q-1)^3\left[3q(q-1)(q-4)+2\right]}}{2}\right|$$

$$=\left(\sqrt{(2q-3)^2(2q-1)^2+8(q-1)^2}\right)(q-1)+\left(4(q-1)^3+1\right).$$

In particular, $GE_{C'}\left(S_2^0\right)=0.6180\times 2+1.6180\times 2=4.472$

and $GE_{C'}\left(S_3^0\right)=\left(\sqrt{15^2+32}\right)\times 2+\sqrt{33^2+512}=2\sqrt{257}+\sqrt{1601}$.

4. Bounds for Minimum Covering Gutman Energy

In this section, we will discuss the bounds of minimum covering Gutman energy. To study the upper bounds of energy of graphs, refer (Liu, 2007). The following lemma is a property of minimum covering Gutman eigenvalues.

Lemma 4.1: For a simple connected graph G with q nodes and p edges and let $C' = \{u_1, u_2, \ldots, u_r\}$ be the covering set, if $\eta_1, \eta_2, ..., \eta_q$ are the minimum covering Gutman eigenvalues obtained from the minimum covering Gutman matrix $A_{C'_g}(G)$, then

$$\sum_{i=1}^{q}\eta_i = \left|C'\right| \quad \text{and} \quad \sum_{i=1}^{q}\eta_i^2 = \left|C'\right| + 2\sum_{1\leq i<j\leq q}\left(d_i d_j d_{ij}\right)^2 \qquad (1)$$

Proof: We have a simple connected (q, p)-graph G. It is well known that the absolute sum of eigenvalues of $A_{C'_g}(G)$ is its trace and sum of squares of eigenvalues of $A_{C'_g}(G)$ is the trace of its square.

That is, $\sum_{i=1}^{q}\eta_i = trace\left(A_{C'_g}(G)\right) = \sum_{i=1}^{q} d_i^{\,2} d_{ii} = \left|C'\right|$

Also, $\sum_{i=1}^{q}\rho_i^{\,2} = trace\left[\left(A_{C'_g}(G)\right)^2\right] = \sum_{i=1}^{q}\sum_{j=1}^{q}(d_i d_j d_{ij})^2 = \sum_{i=1}^{q} d_i^{\,2}(d_{ii})^2 + \sum_{i\neq j}\left(d_i d_j d_{ij}\right)^2$

This implies $\sum_{i=1}^{q}\rho_i^{\,2}=|C'|+2\sum_{1\le i<j\le q}(d_id_jd_{ij})^2$.

Theorem 4.1: For a connected (q, p)-graph with $GE_{C'}(G)$ as minimum covering Gutman energy, $\sqrt{|C'|+2\sum_{1\le i<j\le q}(d_id_jd_{ij})^2}\le GE_{C'}(G)\le\sqrt{q\left(|C'|+2\sum_{1\le i<j\le q}(d_id_jd_{ij})^2\right)}$,

where $|C'|$ is the cardinality of minimum covering set C' of G.

Proof: We have a connected (q, p)-graph G with minimum covering Gutman energy $GE_{C'}(G)$. Also, let C' be the minimum covering set of G.

Claim 1: To obtain the upper bound.

Consider Cauchy-Schwartz inequality $\left(\sum_{i=1}^{q}x_iy_i\right)^2\le\left(\sum_{i=1}^{q}x_i^{\,2}\right)\left(\sum_{i=1}^{q}y_i^{\,2}\right)$.

Take $x_i=1$ and $y_i=|\eta_i|$.

Consequently, $\left(\sum_{i=1}^{q}|\eta_i|\right)^2\le\sum_{i=1}^{q}1\sum_{i=1}^{q}|\eta_i|^2$. This exactly gives $\left(\sum_{i=1}^{q}|\eta_i|\right)^2\le q\sum_{i=1}^{q}\eta_i^{\,2}$.

Hence $(GE_{C'}(G))^2\le q\left(|C'|+2\sum_{1\le i<j\le q}(d_id_jd_{ij})^2\right)\Rightarrow GE_{C'}(G)\le\sqrt{q\left(|C'|+2\sum_{1\le i<j\le q}(d_id_jd_{ij})^2\right)}$ (2)

Claim 2: To obtain the lower bound

However, $(GE_{C'}(G))^2=\left(\sum_{i=1}^{q}|\eta_i|\right)^2\ge\sum_{i=1}^{q}\eta_i^{\,2}=|C'|+2\sum_{1\le i<j\le q}(d_id_jd_{ij})^2$.

So, $GE_{C'}(G)\ge\sqrt{|C'|+2\sum_{1\le i<j\le q}(d_id_jd_{ij})^2}$ (3)

Combining (2) and (3) gives the result.

Corollary 4.1: For a connected (q, p) graph, we have $GE_{C'}(G)\ge\sqrt{|C'|+2q(q-1)}$, $q\ge 2$.

Proof: Obviously, $d_id_jd_{ij}\ge 1$, $\forall\, i\ne j$. Since there are $\frac{q(q-1)}{2}$ pairs of nodes in G, it is clear from (3) of theorem 4.1 that $GE_{C'}(G)\ge\sqrt{|C'|+2\sum_{1\le i<j\le q}(d_id_jd_{ij})^2}$

$$\geq \sqrt{\left|C'\right| + 2 \cdot \frac{q(q-1)}{2}} = \sqrt{\left|C'\right| + q(q-1)},\ q \geq 2\,.$$

Theorem 4.2: For a simple connected $(q,\ p)$ graph G with Δ as the absolute value of the determinant of the minimum covering Gutman matrix $A_{C'_g}(G)$. Then

$\sqrt{\left|C'\right| + 2\sum_{1\leq i<j\leq q}\left(d_i d_j d_{ij}\right)^2 + q(q-1)\Delta^{2/q}} \leq GE_{C'}(G) \leq \sqrt{\left|C'\right| + 2q\sum_{1\leq i<j\leq q}\left(d_i d_j d_{ij}\right)^2}$, where $\left|C'\right|$ is the cardinality of minimum covering set C' of G.

Proof: From the definition of minimum covering Gutman energy of graph and lemma 4.1,

$$\left(GE_{C'}(G)\right)^2 = \sum_{i=1}^{q}\eta_i^{\,2} = \sum_{i=1}^{q}\eta_i^{\,2} + 2\sum_{1\leq i<j\leq q}\left|\eta_i\right|\left|\eta_j\right| = \left|C'\right| + 2\sum_{1\leq i<j\leq q}\left(d_i d_j d_{ij}\right)^2 + \sum_{i\neq j}\left|\eta_i\right|\left|\eta_j\right|.$$

The arithmetic mean is greater than or equal to the geometric mean, for non-negative numbers.

Therefore, $\displaystyle \frac{1}{q(q-1)}\sum_{i\neq j}\left|\eta_i\right|\left|\eta_j\right| \geq \prod_{i\neq j}\left(\left|\eta_i\right|\left|\eta_j\right|\right)^{1/q(q-1)} = \left(\prod_{i=1}^{q}\left|\rho_i\right|^{2(q-1)}\right)^{1/q(q-1)} = \left(\prod_{i=1}^{q}\left|\rho_i\right|\right)^{2/q} = \Delta^{2/q}\,.$

That is, $$\sum_{i\neq j}\left|\eta_i\right|\left|\eta_j\right| \geq q(q-1)\Delta^{2/q}\,.$$

Hence $$\left(GE_{C'}(G)\right)^2 \geq \left|C'\right| + 2\sum_{1\leq i<j\leq q}\left(d_i d_j d_{ij}\right)^2 + q(q-1)\Delta^{2/q}\,. \tag{4}$$

Taking the square root of (4) and combining with the upper bound of theorem 4.1 gives the result.

Theorem 4.3: For a connected $(q,\ p)$ graph G with minimum covering set C' and $GE_{C'}(G)$ as mimimum covering Gutman energy, then

$$GE_{C'}(G) \leq \frac{\left|C'\right| + 2\sum_{1\leq i<j\leq q}\left(d_i d_j d_{ij}\right)^2}{q} + \sqrt{(q-1)\left(\left|C'\right| + 2\sum_{1\leq i<j\leq q}\left(d_i d_j d_{ij}\right)^2 - \left(\frac{\left|C'\right| + 2\sum_{1\leq i<j\leq q}\left(d_i d_j d_{ij}\right)^2}{q}\right)^2\right)}$$

Proof: Applying Cauchy-Schwartz inequality to vectors $(\underbrace{1,1,\ldots,1}_{(q-1)})$ and $(\underbrace{\left|\eta_2\right|,\left|\eta_3\right|,\ldots,\left|\eta_n\right|}_{(q-1)})$,

we obtain $$\left(\sum_{i=2}^{q}\left|\eta_i\right|\right)^2 \leq (q-1)\sum_{i=2}^{q}\eta_i^{\,2}$$

$$\left(GE_{C'}(G)-|\eta_1|\right)^2 \le (q-1)\left(|C'|+2\sum_{1\le i<j\le q}\left(d_i d_j d_{ij}\right)^2-\eta_1^{\,2}\right)$$

$$GE_{C'}(G)\le|\eta_1|+\sqrt{(q-1)\left(|C'|+2\sum_{1\le i<j\le q}\left(d_i d_j d_{ij}\right)^2-\eta_1^{\,2}\right)} \tag{5}$$

Now, define a function
$$f(x)=x+\sqrt{(q-1)\left(|C'|+2\sum_{1\le i<j\le q}\left(d_i d_j d_{ij}\right)^2-\eta_1^{\,2}\right)} \tag{6}$$

We set $\eta_1 = x$ for $\eta_1 \ge 1$

Taking square, $x^2=\eta_1^{\,2}\le\sum_{i=1}^{n}\eta_1^2=|C'|+2\sum_{1\le i<j\le q}\left(d_i d_j d_{ij}\right)^2$. This implies $x\le\sqrt{|C'|+2\sum_{1\le i<j\le q}\left(d_i d_j d_{ij}\right)^2}$.

Again, taking the derivative $f'(x)=0$ gives $x=\sqrt{\dfrac{|C'|+2\sum_{1\le i<j\le q}\left(d_i d_j d_{ij}\right)^2}{q}}$.

This gives a decreasing function $f(x)$ in the interval

$$\sqrt{\frac{|C'|+2\sum_{1\le i<j\le q}\left(d_i d_j d_{ij}\right)^2}{q}}\le x\le\sqrt{|C'|+2\sum_{1\le i<j\le q}\left(d_i d_j d_{ij}\right)^2}$$

and
$$\sqrt{\frac{|C'|+2\sum_{1\le i<j\le q}\left(d_i d_j d_{ij}\right)^2}{q}}\le\frac{|C'|+2\sum_{1\le i<j\le q}\left(d_i d_j d_{ij}\right)^2}{q}\le\eta_1.$$

Therefore, $f(\eta_1)\le f\left(\dfrac{|C'|+2\sum_{1\le i<j\le q}\left(d_i d_j d_{ij}\right)^2}{q}\right)$ and our result follows.

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