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EFFECT OF GUP ON THE RADIAL COMPONENT OF THE KINETIC ENERGY

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Abstract

Various theories of quantum gravity predict modifications of the Heisenberg uncertainty principle near the Planck scale, known as the generalized uncertainty principle (GUP). In this work, we study the effects of GUP on the equation of motion of a particle. Here, GUP preserves the rotational symmetry of the space-time. Then, considering the Kepler potential, we investigate the orbit motion of a particle and obtain the contribution of the radial component of the kinetic energy in this model.

Keywords

Generalized uncertainty principle; Equation of motion; Kepler potential

1. Introduction

Heisenberg obtained the Uncertainty Principle on very general grounds, using only the quantization of the electromagnetic radiation field. He did not consider gravitational effects in his uncertainty relation, as are usually assumed to be negligible. However, at increasingly large energies the gravitational interaction is more and more important.

Various approaches to quantum gravity (such as string theory, doubly special relativity theories, as well as black hole physics) suggest that near the Planck scale, the Heisenberg Uncertainty Principle should be modified. The modified Uncertainty Principle is called Generalized Uncertainty Principle (GUP) (Amati et al., 1989; Maggiore, 1994; Garay, 1995). An important consequence of quantum gravity scenarios is the existence of a minimum length. Because the Heisenberg uncertainty principle does not exert any restriction on the measurement precision of the particle's positions or momentum, there is no minimum measurable length in the usual Heisenberg picture.

In the past few years, quantum mechanics with modification of the usual canonical commutation relations has been investigated intensively (Kempf et al., 1995; Magguchi & Smolin, 2002 and 2005; Das & Vagenas, 2008; Ali et al., 2009; Pedram, 2010). Such works, which are motivated by string theory and quantum gravity, suggest a minimal length as $\Delta x \geq \hbar\sqrt{\beta}$. This means that there is no possibility to measure coordinate x with accuracy smaller than $\hbar\sqrt{\beta}$, where β is a positive parameter and depends on the expectation value of the position and momentum operators (Garay, 1995; Kempf et al., 1995). Because in string theory the minimum observable distance is the string length, $\sqrt{\beta}$ is proportional to this length. If we set $\beta=0$ the usual Heisenberg algebra is recovered. In the low energy limit, these quantum gravity effects can be neglected. But in circumstances like the very early universe or in the strong gravitational field of a black hole, one has to consider these effects. The modification induced by the generalized uncertainty principle (GUP) on the classical orbits of particles in a central force potential has been considered in ref. (Benezik et al., 2002; Ahmadi & Khodagholizadeh, 2014).

In this paper, we are going to proceed to study the effects of GUP on the equation of motion of a particle in a Kepler potential field. Then, we will compare the contribution of the radial component of kinetic energy in this model with that of the ordinary classical mechanics.

2. The modified equation of motion

Let us start by considering the commutation relation with the GUP as:

$$[x_i, p_j] = i\hbar (1 + \beta p^2) \delta_{ij} \quad (1)$$

Where β is a positive parameter that depends on the expectation value of x_i and p_j which are position and momentum components respectively. We may assume the components of the momentum p_j commute with each other:

$$[p_i, p_j] = 0 \quad (2)$$

So the commutation relations of coordinates become

$$[x_i, x_j] = 2i\hbar\beta (p_{ix}j - p_{jxi}) \quad (3)$$

Since commutation relations in (1), (2) and (3) do not break the rotational symmetry, we are able to express the rotation generators in terms of the position and momentum operators as follows:

$$L_{ij} = 1 + \beta p^2 (x_{ip}j - x_{jpi}) \quad (4)$$

In the classical limit, using the Poisson bracket, the relations 1, 2 and 3 can be written as

$$\{x_i, p_j\} = (1 + \beta p^2) \delta_{ij} \quad (5)$$

$$\{p_i, p_j\} = 0 \quad (6)$$

$$\{x_i, x_j\} = 2\beta (p_{ix}j - p_{jxi}) \quad (7)$$

The Poisson bracket must possess the same properties as the quantum mechanics commutators, namely, it must be ant symmetric, bilinear, and satisfy the Leibniz rules and the Jacobi identity. So, we can find the general form of the Poisson bracket for any functions of the coordinates and moment as follows:

$$\{F, G\} = \left(\frac{\partial F}{\partial x_i} \frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial x_j} \right) \{x_i, p_j\} + \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j} \{x_i, x_j\} \quad (8)$$

Now, to study the motion of a macroscopic object in a central potential, we write the Hamiltonian as

$$H = \frac{p_i^2}{2m} + v(r) \quad (9)$$

The time evolution of the coordinates and moment become:

$$\dot{x}_i = \{x_i, H\} = \{x_i, p_j\} \frac{\partial H}{\partial p_j} + \{x_i, x_j\} \frac{\partial H}{\partial x_j} \quad (10)$$

$$\dot{p}_i = \{p_i, H\} = -\{x_i, p_j\} \frac{\partial H}{\partial x_j} + \{p_i, p_j\} \frac{\partial H}{\partial p_j} \quad (11)$$

Using equations 5, 6, 7 and neglecting terms with order of β^2 , we have:

$$\dot{x}_i = (1 + \beta p^2) \frac{p_i}{m} - 2\beta \left(\frac{1}{r} \frac{\partial v}{\partial r} \right) L_{ij} x_j \quad (12)$$

$$\dot{p}_i = -(1 + \beta p^2) x_i \left(\frac{1}{r} \frac{\partial v}{\partial r} \right) \quad (13)$$

For motion in central potential field, the generators of rotations L_{ij} are conserved due to

$\{L_{ij}, H\} = 0$. The conservation of the L_{ij} imply that the motion of the particle will be confined to two dimensional plane spanned by the coordinate and momentum vectors ant any point in time. Finally, using the above equations, we can obtain the equations of motion of a particle as follows:

$$m\ddot{x}_i = (1 + 3\beta p^2)\dot{p}_i - 2\beta \left(\frac{1}{r} \frac{dv}{dr}\right) p_i \quad (14)$$

3. The Kepler potential field

Here, using the Kepler potential $V(r) = -k/r$, equation (14) becomes as:

$$m\ddot{x}_i = (1 + 4\beta p^2) \frac{x_i}{r} \left(\frac{k}{r^2}\right) - 2\beta \frac{k}{r^3} p_i \quad (15)$$

Since β is a very small parameter and r is very large, we can ignore the second term in comparison with the first term. Now, we rewrite equations (15) in spherical coordinates:

$$m(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2\theta) = -[1 + 4\beta m^2(\dot{r}^2 + r\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2\theta)] \frac{\partial V}{\partial r} \quad (16)$$

$$m \frac{d}{dt}(r^2\dot{\theta}) - mr^2\dot{\phi}^2 \sin\theta \cos\theta = 0 \quad (17)$$

$$\frac{d}{dt}(mr^2\dot{\phi} \sin^2\theta) = 0 \quad (18)$$

Since the generators of rotations L_{ij} are conserving, the case of a plane orbit is a valid solution for above equations. For simplicity, we consider the equatorial orbits (*i.e.*, $\theta = \pi/2$) and obtain these equations

$$m(\ddot{r} - r\dot{\phi}^2) = -\frac{k}{r^2} - 4\beta m^2(\dot{r}^2 + r^2\dot{\phi}^2) \frac{k}{r^2} \quad (19)$$

$$m \frac{d}{dt}(r^2\dot{\theta}) = 0 \quad (20)$$

$$\frac{d}{dt}(mr^2\dot{\phi}) = 0 \quad (21)$$

Using equation (21), we can have $L = mr^2\dot{\phi}$ which is a constant. By substituting in equation (19), we obtain:

$$m\ddot{r} = \frac{L^2}{mr^3} \left(1 - \frac{4\beta mk}{r}\right) - \frac{k}{r^2} - 4\beta km^2 \frac{\dot{r}^2}{r^2} \quad (22)$$

Now, we can consider a new potential for the right- hand side of above equation

$$V = \frac{L^2}{2mr^2} - \frac{4\beta k L^2}{3 r^3} - \frac{k}{r} - 4\beta km^2 \frac{\dot{r}^2}{r^2} = \frac{L^2}{2mr^2} \left(1 - \frac{8m\beta k}{3 r}\right) - \frac{k}{r} - 4\beta km^2 \frac{\dot{r}^2}{r^2} \quad (23)$$

The second term in parentheses of the above equation is very small compared to one, and then we can neglect it. In following we are going to concentrate on the energy of the system, which is a composition of kinetic and potential energy. As we know

$$E = T + V = \frac{1}{2} \left(1 - \frac{8\beta mk}{r}\right) m\dot{r}^2 + \frac{1}{2} mr^2 \dot{\phi}^2 - \frac{k}{r} \quad (24)$$

We see that the contribution of the radial components of kinetic energy is less than ordinary classical mechanics. Also, if we consider object with $E \geq 0$, in $r = 8\beta mk$, we have $mr^2 \dot{\phi}^2 = \frac{1}{4\beta}$.

By assuming $x_{min} = \hbar \sqrt{\beta}$, we obtain

$$P = \frac{\hbar}{2x_{min}} \quad (25)$$

Because $p_{max} x_{min} = \hbar/2$, from above equation, we can conclude $P = p_{max}$ and $8\beta mk = x_{min}$.

Then we can obtain

$$\beta = \frac{\hbar^2}{64m^2k^2} \quad (26)$$

Where β depends on the mass of object (m) and the constant of potential field (k). We see, whatever m and K are larger, β become smaller.

4. Conclusion

In this paper, we have investigated the effects of GUP on the equation of motion of a particle. Also, we have studied the equation of motion of a particle in a Kepler potential field. We found that the contribution of the radial components of kinetic energy is less than ordinary

classical mechanics. Finally, in a dynamic system, we obtained a relation between the parameter, the mass of the moving object m and the constant of the Keplerpotential K . This is the key point of this paper.

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